Algebraic relaxation of an order parameter in randomly coupled limit-cycle oscillators

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In their recent paper [Phys. Rev. E **58**, 1789 (1998)], Stiller and Radons (SR) study, following our earlier work [Phys. Rev. Lett. **68**, 1073 (1992)], the behavior of globally and randomly coupled phase oscillators with distributed intrinsic frequencies. They claim that their simulation results do not confirm the power-law behavior of an order parameter found numerically by the author, attributing its cause to the poor precision of the author's integration scheme. Here demonstrated is that the power law survives even for a scheme better than SR's, provided that finite-size effects are properly taken into account, as was done in our previous work.

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The behavior of large populations of coupled nonlinear oscillators is now one of the central subjects in nonlinear dynamics [1–3]. Although it is usually studied for the case of nonrandom interactions, the architecture of coupling in any kind of real coupled-oscillator systems should involve more or less quenched disorder. If such disorder is weak enough, then it will not cause any significant change in the system's behavior. However, one may expect the emergence of qualitatively new features when randomness in coupling is not decoratively feeble. For example, many biological and physiological oscillator systems, including the brain, might be examples of such a case [1,4]. It is therefore an important and interesting subject to examine the behavior of randomly coupled oscillators. From this point of view, the author started investigations into pools of randomly coupled limit-cycle oscillators more than a decade ago [5] and later proposed a model of ''oscillator glass'' [6],

$$\dot{\theta}_j=\Omega_j+(2\pi)^{-1}\sum_{i=1}^{N} J_{ij}\sin(\theta_i-\theta_j) \quad (j=1,\ldots,N), \tag{1}$$

where the overdot means the time derivative; θ_j is the phase of oscillator j, Ω_j being its intrinsic frequency, which is assumed to be distributed within the population with density $f(\Omega)$, and $J_{ij}=J_{ji}$ are independent random coefficients obeying the normal distribution $g(J_{ij})=(2\pi J^2/N)^{-1/2}\exp(-NJ_{ij}^2/2J^2)$; J is the control parameter [7]. This model is analogous to the Sherrington-Kirkpatrick model of spin glasses [8], characterized by not only randomness but also strong frustration. Of several results in Ref. [6], what is taken up here is the behavior of $Z(t)$ defined by

$$Z(t)=\frac{1}{N}\sum_{j=1}^{N}\exp\{i\theta_j(t)\} \quad (i\equiv\sqrt{-1}),$$

which is a standard order parameter in the study of coupled limit-cycle oscillators [2,3]. In Ref. [6], the intrinsic frequencies are set so as to obey $f(\Omega)=(2\pi)^{-1/2}\exp(-\Omega^2/2)$ as an example (see below for details) [9]. What is reported there [see in particular Fig. 5(a) therein [10]] is that the decay of $|[Z(t)]|$ for t large changes from exponential to algebraic as

$$|[Z(t)]|\sim t^{-\alpha} \tag{2}$$

when J exceeds J_c, a threshold value, where $[Z(t)]$ stands for an average of $Z(t)$ over a number of different realizations of J_{ij}, which average will hereafter be referred to as a ''sample-average'' following Refs. [6,11] and the number of realizations used will be denoted by N_s [12]. In that work, numerical integration was performed with the Euler scheme of time step $\Delta t=2\pi\times 0.01$ for $N=500,1000,2000$ and $N_s=10$. For the larger values of N, J_c is near 6, and the exponent α depends on J [see Fig. 5(b) of Ref. [6]].

Recently, however, Stiller and Radons (SR) [13] have reported that their results are essentially different: they find exponential decay for $J<24.5$ and algebraic decay only for $J=24.5$, leaving the region $J>24.5$ unsettled because of the complex behavior of the order parameter therein. Their integration scheme is the Heun method with $\Delta t=2\pi\times 10^{-3}$, which is expected to have better accuracy than the author's [6]. From this fact, they suggest that the slow relaxation found by the author is not correct, being a discretization effect due to the low order integration scheme. The main purpose of this paper is to point out the following: (1) An alternative computation based on a higher-order method which is expected to be more accurate than SR's supports the algebraic decay; (2) what is crucial is finite-size effects, which Stiller and Radons do not seem to take into account.

Although not discussed here, there are other and recent studies of randomly coupled oscillators (see, e.g., papers in [14]).

Our comments on the relevant part of the SR paper [13] are as follows.

(1) As a demonstration of the discretization effect, they compute the Lyapunov exponent with both their method and ours in Ref. [6] to compare the results [Fig. 4 of Ref. [13]]. However, this does not make sense because the problem is the relaxation of Z and NOT the behavior of the Lyapunov exponent. The integration error caused by a low-order scheme may accumulate to have a serious influence on the final result when the integration is made over a long time interval, just as in the calculation of the Lyapunov exponent, but the situation can be different with calculations over a relatively short time interval. In the present case, the time span concerned is order of $2\pi\times 1$. For such a short period, it is possible that even low order schemes produce sufficiently accurate results. Our new calculations described below will show that this is indeed the case.

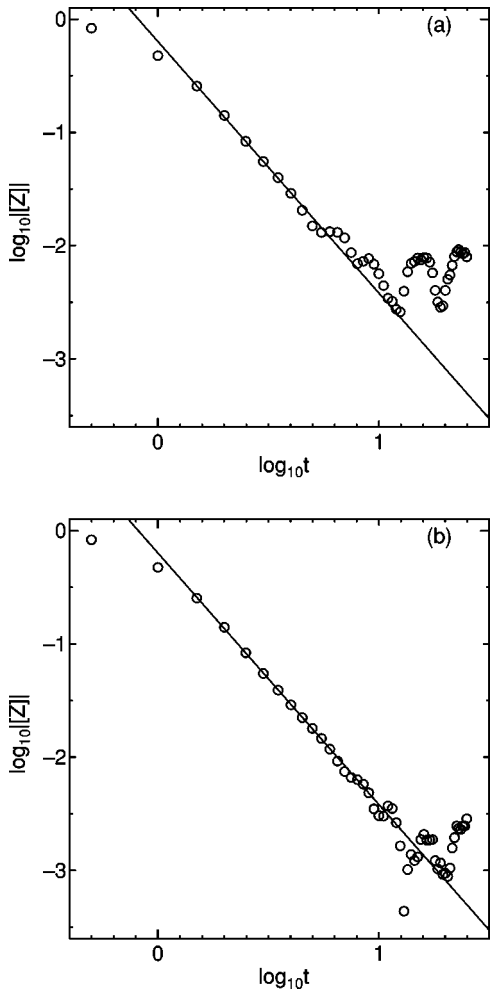

FIG. 1. Behavior of the sample-averaged order parameter for $J=10$: (a) $N=1000$, $N_s=50$; (b) $N=4000$, $N_s=52$.

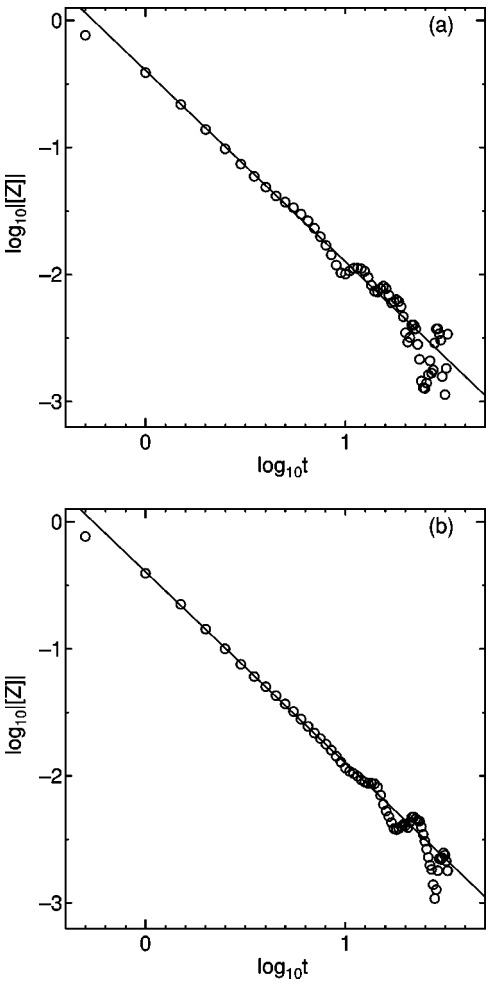

FIG. 2. Behavior of the sample-averaged order parameter for $J=15$: (a) $N=1000$, $N_s=50$; (b) $N=4000$, $N_s=50$.

(2) The most crucial point is the check of finite-size effects. The paper by SR does not mention anything on this important point, reporting only the result for $N=1000$. Any finite-size system cannot escape from finite-size effects, one of which is sample-dependence: the smaller N, the larger this dependence. By this reason, sample-averaging as mentioned above is indispensable to obtain a meaningful result for N finite. When N is not large enough, it is not surprising even if a result for one particular sample largely deviates from the infinite-size behavior. Another finite-size effect manifests itself in the long time behavior of the order parameter, i.e., it does not vanish even in the limit $t\to\infty$, but keeps fluctuating with a small amplitude of order $N^{-1/2}$ [15]. This means that the true infinite-size behavior can only be expected to appear while the value of the order parameter stays well above the level below which such a finite-size behavior starts to appear. In view of these, in the author's previous work, three values of N, as already recalled, were taken to confirm that increasing N resulted in a longer power-law range of t.

In order to verify that the algebraic decay of the order parameter is not a false one created by low accuracy of the integration scheme, new calculations have been done for $N=1000$ and 4000 using a Runge-Kutta-Gill routine with $\Delta t=0.01$. The Heun method adopted by SR is of second order while the present is of fourth order. The latter for $\Delta t=0.01$ should therefore have better accuracy than the former for $\Delta t=2\pi\times10^{-3}$, which is SR's choice. As in the work of Ref. [6], intrinsic frequencies have been prepared as follows:

$$\Omega_{(N/2)+k}=(x_{k-1}+x_k)/2, \quad \Omega_{(N/2)-k+1}=-(x_{k-1}+x_k)/2 \tag{3}$$

for $k=1,\ldots,N/2$, where x_k are recursively defined by

$$x_{k+1}=x_k+N^{-1}/f(x_k)$$

starting from $x_0=0$. The symmetric coupling coefficients J_{ij} have been produced by a Gaussian random number generator. Note that by these settings, the sum $\Sigma_{j=1}^{N}\theta_j$ is exactly conserved as in the infinite-size system. The initial condition is $\theta_j=0$ for all j. Since very expensive (in particular for $N=4000$), the computation has been limited to $J=10$ and 15 here, which values are above the threshold J_c reported previously by the author (thus lying in the power law region) and well below SR's threshold (thus in the exponential region according to SR). Figures 1 and 2 show double logarithmic plots of $|[Z(t)]|$ vs t for $J=10$ and 15, respectively. Increasing N_s tends to make the power law fit better. To check the pure effect of increasing N, however, the comparison between $N=1000$ and 4000 is made for exactly or nearly the same N_s. Let us first check the results for $N=1000$ dis-

played in (a) of each figure, where the fit line for the data in (b) is copied to make comparison to (b) easier. The common feature is that except for an initial transient period, an algebraic decay begins to last until it is influenced and then interrupted by irregular oscillations, one of the finite-size effects mentioned above. It is also seen that the algebraic decay begins earlier for J larger. Let us then go on to (b) of each figure where the result for $N=4000$ is displayed. As is expected, for both values of J, the time interval in which an excellent power-law fit is possible is found to substantially expand towards the large t side and hence the appearance of the irregular oscillations to be delayed accordingly. This fact is clear evidence that the deviation from the power law for t large is nothing but a finite-size effect. As a further check, the values of exponent α for $J=10$ and 15 were determined by a least squares fit to compare to those due to the previous method. For $J=10$, the work of Ref. [6] ($N=1000$) yielded $\alpha=2.277\pm0.004$ ($1.76\leqslant t\leqslant 4.27$), while the present work ($N=4000$) led to $\alpha=2.220\pm0.007$ ($1.5\leqslant t\leqslant 6$). For $J=15$, no data are available from the previous work; a new computation with the same method for the same N and N_s as in Ref. [6] resulted in $\alpha=1.599\pm0.005$ ($1.13\leqslant t\leqslant 3.52$), while its counterpart here($N=4000$) is $\alpha=1.506\pm0.006$ ($1.0\leqslant t\leqslant 9.5$). These results demonstrate that the computational setting adopted in Ref. [6] actually worked sufficiently well.

To conclude, the algebraic decay of the order parameter reported previously has been reconfirmed by adopting a more accurate algorithm for integration and by making both the system size and the number of samples much larger. This result does not agree with Stiller and Radons' claim that the decay of the order parameter remains exponential until J reaches 24.5 and their other claim that the nonexponential relaxation reported in Ref. [6] is a result of poor accuracy of the integration scheme adopted there. Although only two values of J have been tested here, we believe that the same conclusion holds for other values of J at least in the range explored in Ref. [6] ($J\leqslant 20$). The key point is to take finite-size effects into account; similar comments as given above might be relevant to another part of Stiller and Radons' work [13]. Admittedly, any numerical approach has its own limits: analytical studies would be indispensable to establish the slow relaxation (2) over a range of the parameter.

Numerical calculations of this work have been performed with Fujitsu VPP 700/56 of The Computer Center, Kyushu University.

[1] A.T. Winfree, *The Geometry of Biological Time* (Springer, New York, 1980).
[2] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).
[3] For a collection of recent papers, see H. Daido Int. J. Bifurcation Chaos Appl. Sci. Eng. **7**, 807 (1997).
[4] A. K. Engel *et al.*, Trends Neurosci. **15**, 218 (1992).
[5] H. Daido, Prog. Theor. Phys. **77**, 622 (1987).
[6] H. Daido, Phys. Rev. Lett. **68**, 1073 (1992).
[7] Note that the original variables in Eq. (2) of Ref. [6] are here transformed as $2\pi\theta\rightarrow\theta$, $2\pi t\rightarrow t$.
[8] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
[9] Similar results were obtained later for a Lorentzian distribution of Ω_j as well [H. Daido (unpublished)].
[10] A correction: The time axis in Fig. 5(a) of Ref. [6] should read $t/0.01$.
[11] Although the notation $[\cdot]$ is not used in Ref. [6], its content is included in the explanation of Fig. 5 (left column, p. 1075) and its caption.
[12] One may consider $[|Z|]$ instead of $|[Z]|$, but by self-averaging, both quantities will probably become equal to each other in the limit $N\rightarrow\infty$. A reason for choosing the latter is that it is expected to be analytically more tractable than the former.
[13] J. C. Stiller and G. Radons, Phys. Rev. E **58**, 1789 (1998).
[14] L. L. Bonilla, C. J. Pérez Vicente, and J. M. Rubí, J. Stat. Phys. **70**, 921 (1993); S. Uchiyama and H. Fujisaka, Phys. Rev. E **56**, 99 (1997); K. Park, S. W. Rhee, and M. Y. Choi, *ibid.* **57**, 5030 (1998); T. Aonishi, K. Kurata, and M. Okada, Phys. Rev. Lett. **82**, 2800 (1999).
[15] H. Daido, J. Stat. Phys. **60**, 753 (1990).